AN IMPLIED INCOME INEQUALITY INDEX USING L_1 NORM ESTIMATION OF LORENZ CURVE

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ABSTRACT

Distribution of income is among the most important issues in welfare economics. It is discussed that for more equal distribution of income and welfare government should redistribute income and to achieve more equality and promote total utility of the society. Economic literature provides different ways to measure income inequality. While there are alternative methods, there is no best way to calculate the inequality index. Most common inequality indices provide information about points on the distribution function and analyze the inequality of income without any reference to the amount of the money needed to improve the distribution.

In this paper, we identify an income inequality index. Using this index, we will estimate the Lorenz Curve function parameters and show how much transfer payment is needed to achieve a desired distribution of income consistent with the perceived economic goals of the society.

Therefore, we design a model to estimate Lorenz Curve and find a fiscal-compensation-based index for reduction of the degree of inequality. By this approach, any Census summary data can be used to measure the distribution of income. Using our calculated implied-inequality-index, we may redistribute a percentage of income to the lower income group and thereby improve the distribution.

Keywords: Income distribution, Inequality index, Lorenz curve.

JEL: MATHEMATICAL AND QUANTITATIVE METHODS (C63)

1. INTRODUCTION

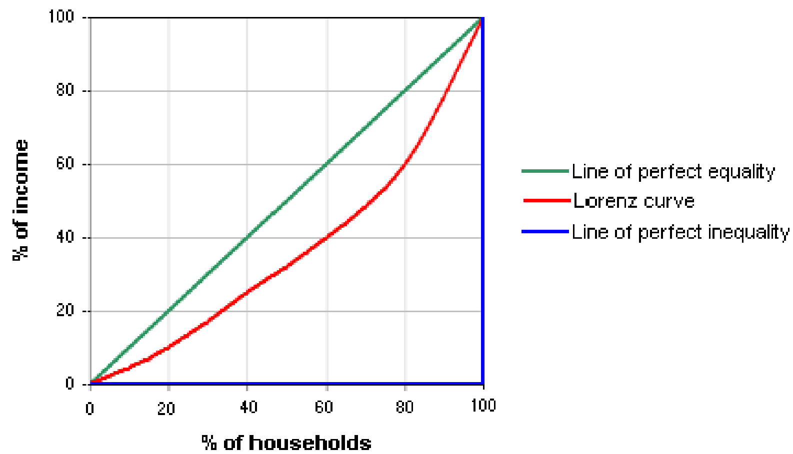
Estimation of the Lorenz curve is a challenge and is coupled with some difficulties. To estimate, first we need to define an appropriate functional form that can accept different curvatures. Moreover, to generate the necessary data set for estimation of the corresponding parameters, a large scale of computation on sample income data is inevitable. In this paper, we introduce a shortcut and use the probability density function of population income to estimate the Lorenz function parameters. The continuous L_1 norm smoothing method will be developed to estimate the regression parameters. We use two different probability density functions: (a) Pareto density distribution function that is integrable and (b) log-normal function that is more suitable for a wider range of income but is not integrable. In the latter case, we need to define a general Lorenz functional form and apply the continuous L_1 norm estimation of linear one and two parameters models for a discrete data to corresponding parameters. The method is applied to estimate the Lorenz functional form proposed by Gupta (1984) and Bidabad and Bidabad (1989). We can use the functional form and parameters of probability distribution function of income to determine the L_1 norm approximation of the corresponding Lorenz curve of the population.

In the next step, an implied inequality index is introduced. Most common inequality indices provide comparative information about the distribution of inequality without any reference to the amount of the money needed to improve the distribution. In this paper, we identify an income inequality index to show how much transfer payment is needed to achieve a desired distribution of income consistent with the perceived economic equality goals of the policy-maker.

2. REVIEW

Distribution of income is central to one of the most enduring issues in economics. The degree of income inequality can be illustrated with a Lorenz curve. The Lorenz curve is a graphical representation of the cumulative income distribution function. It shows what portion of the total income y is received by the bottom percentages of the households. The percentages of households are plotted on horizontal axis, and the percentages of earned incomes are plotted on the vertical axis as figure 1 shows.

Figure 1: Lorenz Curve



The income inequality shown by the Lorenz curve can be measured by Gini ratio. This coefficient is a measure to express distribution inequality. It is defined as a ratio between 0 and 1. Its numerator is the area between the Lorenz curve and the diagonal line, which is the uniform distribution line; and the denominator is the area under the uniform distribution line. Since the skewness of income distribution is persistently exhibited for different populations, the Lorenz curve becomes a method to analyze the skew distributions. It is discussed that Pearsonian family distributions are rival functions to explain income distribution. There is also a relation between the area under the Lorenz curve and the corresponding probability distribution function (see, Kendall and Stuart (1977)). That is, when the probability distribution function is known, we may find the corresponding Lorenz curve and Gini coefficient as well.
Estimation of the Lorenz curve is confronted with some difficulties. For this estimation, we should define an appropriate functional form that can accept different curvatures. There is another problem, that is, to create large-size dataset for estimating the corresponding parameters of the Lorenz curve, a large amount of computation on raw sample income data is inevitable. Obviously, these problems despite of their computational difficulties make the significance of the estimated parameters poor (see, Bidabad and Bidabad (1989)). To avoid this, we try to estimate the functional form of the Lorenz curve by using continuous information.
In this paper, we use the probability density function of population income to estimate the Lorenz function parameters. The continuous L_1 norm smoothing method, which will be developed for estimating the regression parameters is used to solve this problem. However, we concentrate on two rival probability density functions of Pareto and log-normal. Since, the former is simply integrable, there is no general problem to derive the corresponding Lorenz function and the function is uniquely derived. However, in

the latter case, the log-normal density function which has better performance for full income range than Pareto distribution (which better fits to higher income range, (see, Cramer (1973), Singh and Maddala (1976), Salem and Mount (1974)), is not integrable and we cannot determine its corresponding Lorenz function. In this regard, we should solve the problem by defining a general Lorenz curve functional form and applying the L_1 norm smoothing to estimate the corresponding parameters.

Thus, the continuous L_1 norm estimation problems of linear one and two parameter models are solved. Bidabad (1988a,b) solved the approximation method for discrete case. Some directions were also proposed by Bidabad (1989a,b) for continuous smoothing case. Now, the method is applied to estimation of the Lorenz curve Gupta (1984) and Bidabad and Bidabad (1989) functional forms.

To have a better understanding and policy arrangements about the inequality of income distribution in a region, it is not enough to know the Lorenz curve and conventional inequality indices. The redistribution policies need to deal with specific budget guidelines to promote the society to a more equal position. Economic literature provides different ways to measure income inequality (Atkinson (1970); Sen (1973); Cowell (1977)). Some of the most commonly used measures include the Gini coefficient; the decile ratio; the proportions of total income earned by the bottom 50%, 60%, and 70% of households; the Robin Hood index; the Atkinson index; and Theil's entropy measure. The Gini is calculated as the ratio of the area between the Lorenz curve and the 45° line, to the whole area below the 45° line. Kakwani (1980) is some recalculation of Gini and measures the length of Lorenz curve. The Robin Hood index is equivalent to the maximum vertical distance between the Lorenz curve and the line of equal incomes. The Atkinson (1970) index is one of the few inequality measures that explicitly incorporate normative judgments about social welfare. It is derived by calculating the so-called equity-sensitive average income, which is defined as that level of per capita income, which if enjoyed by everybody would make total welfare exactly equal to the total welfare generated by the actual income distribution. Theil (1967) entropy measure derives from the notion of entropy in information theory.

Obviously, there is no single "best" measure of income inequality. Some measures (e.g., the Atkinson index) are more bottom-sensitive than others are; i.e. more strongly correlated with the extent of poverty. The measures perform differently under various types of income transfers. For instance, the Gini is much less sensitive to income transfers between households if they lie near the middle of the income distribution compared to the tails. The Robin Hood index is insensitive with respect to income transfers between households on the same side of the mean income, and so on. While there are alternative methods, there is no best way to calculate the inequality index specially concentrating on fiscal view. That is they generally analyze the distribution without inferring about the amount of fund needed to correct income inequality.

The viewpoint of this paper is to introduce an implied inequality index, which satisfies these policy implications needs. In the last part of this paper, we design an index that can be used for reduction of the degree of inequality. We show how to use any income census summary data (i.e. the average income and the median) to measure the distribution of income and calculate the amount of money needed to be levied on rich and then transferred to the poor to promote income distribution of the society.

2. L_1 NORM OF CONTINUOUS FUNCTIONS

Generally, L_p norm of a function f(x) (see, Rice and White (1964)) is defined by,

$$\|f(x)\|_p == \int_{x\in I}\left(|f(x)|^P dx\right)^{\frac{1}{p}} \tag{1}$$

Where, "I" is a closed bounded set. The L_1 norm of f(x) is simply written as,

$$\|f(x)\|_1 = \int_{x\varepsilon I} |f(x)|dx \tag{2}$$

Suppose, the non-stochastic function $f(x,\boldsymbol{\beta})$ and the stochastic disturbance term $\underline{u}$ form y(x) as follows,

$$y(x) = f(x, \boldsymbol{\beta}) + u \tag{3}$$

Where, $\boldsymbol{\beta}$ is unknown parameters vector. Rewriting u as the residual of y(x)-f(x,$\boldsymbol{\beta}$), for L_1 norm approximation of "$\boldsymbol{\beta}$" we should find "$\boldsymbol{\beta}$" vector such that the L_1 norm of "u" is minimum. That is,

$$\underset{\boldsymbol{\beta}}{\text{Min:}}\ S = \|u\|_1 = \|y(x)-f(x,\boldsymbol{\beta})\|_1 = \int_{x\varepsilon I} |y(x)-f(x,\boldsymbol{\beta})|dx \qquad (4)$$

3. LINEAR ONE PARAMETER L_1 NORM CONTINUOUS SMOOTHING

Redefine f(x,$\boldsymbol{\beta}$) as βx and y(x) as the following linear function,

$$y(x) = \beta x + u \qquad (5)$$

Where, "$\boldsymbol{\beta}$" is a single (non-vector) parameter. Expression (4) reduces to:

$$\underset{\beta}{\text{Min:}}\ S = \|u\|_1 = \|y(x)- \beta x\|_1 = \int_{x\varepsilon I} |y(x)- \beta x|dx \qquad (6)$$

The discrete analogue of (6) is solved by Bidabad (1988a,89a,b). In those papers, we proposed applying discrete and regular derivatives to the discrete problem by using a slack variable "t" as a point to distinguish negative and positive residuals. A similar approach is used here to minimize (6). To do so in this case certain Lipschitz conditions are imposed on the functions involved (see, Usow (1967a)). Rewrite (6) as follows:

$$\underset{\beta}{\text{Min:}}\ S = \int_{x\varepsilon I} |x|\ \ |y(x)/x - \beta|dx \qquad (7)$$

Let's define "I" as a closed interval [0,1]. The procedure may be applied to other intervals with no major problem (see, Usow (1967a), Hobby and Rice (1965), Kripke and Rivlin (1965)). To minimize this function we should first remove the absolute value sign of the expression after the integral sign. Since "x" belongs to a closed interval "I", both functions, y(x) (which is a linear function of "x") and y(x)/x are smooth and continuous. And since y(x)/x is uniformly increasing or decreasing function of "x", a value of tεI can be found to have the following properties,

$$\begin{aligned} y(x)/x &< \beta && \text{if } x < t \\ y(x)/x &= \beta && \text{if } x = t \\ y(x)/x &> \beta && \text{if } x > t \end{aligned} \qquad (8)$$

Value of the slack variable "t" actually is the border of negative and positive residuals. If value of "t" were known, when x=t we could calculate optimal value of "β". Nevertheless, nor "t" neither "β" are known. To solve, according to (8), we can rewrite (7) as two separate definite integrals with different upper and lower bounds.

$$\underset{\beta}{Min}\ \ S = - \int_0^t |x|\ (y(x)/x - \beta)dx + \int_t^1 |x|\ (y(x)/x - \beta)dx \qquad (9)$$

Decomposition of (7) into (8) has been done by use of the slack variable "t". Since both "β" and "t" are unknown, to solve (9), we partially differentiate it with respect to "t" and "β".

$$\frac{\partial S}{\partial \beta} = \int_0^t |x|dx - \int_t^1 |x|dx = 0 \qquad (10)$$

and using Liebniz rule to differentiate the integrals with respect to their variable bounds "t", yields,

$$\frac{\partial S}{\partial t} = -|t|\left[\frac{y(t)}{t} - \beta\right] - |t|\left[\frac{y(t)}{t} - \beta\right] = 0 \qquad (11)$$

Since "x" belongs to [0,1], equation (10) can be written as,

$$\int_0^t xdx - \int_t^1 xdx = 0 \qquad (12)$$

Or,

$$\tfrac{1}{2} t^2 - \tfrac{1}{2} + \tfrac{1}{2}t^2 = 0 \qquad (13)$$

Which yields,

$$t = \sqrt{2}/2 \qquad (14)$$

Substitute for "t" in equation (11), yields,

$$\beta = \frac{y(\sqrt{2}/2)}{\sqrt{2}/2} \tag{15}$$

Given that y(t) is function y(x) evaluated at x=t. Value of "β" given by (15) is the optimal solution of (6). The above procedure in fact is generalization of Laplace weighted median for continuous case. Before applying this to Lorenz curve, let us develop the procedure for the two parameters linear model.

4. LINEAR TWO PARAMETERS L_1 NORM CONTINUOUS SMOOTHING

To apply the above technique to the linear two parameters model, rewrite (4) as,

$$\min_{\alpha,\beta}: S=\|u\|_1=\|y(x)-\alpha-\beta x\|_1=\int_{x\varepsilon I} |y(x)-\alpha-\beta x|dx \tag{16}$$

Where, "α" and "β" are two single (non-vector) unknown parameters and y(x) and "x" are as before. According to Rice (1964c), let f(α*,β*,x) interpolates y(x) at the set of canonical points $\{x_i;i=1,2\}$, if y(x) is such that: y(x)-f(α*,β*,x) changes sign at these x_i's and at no other points in [0,1], then f(α*,β*,x) is the best L_1 norm approximation to y(x) (see also, Usow (1967a)). With the help of this rule, if we denote these two points to t_1 and t_2 we can rewrite (16) for I=[0,1] as,

$$S=\int_0^{t_1} [y(x)-\alpha-\beta x]dx - \int_{t_1}^{t_2} [y(x)-\alpha-\beta x]\, dx + \int_{t_2}^{1} [y(x)-\alpha-\beta x]\, dx \tag{17}$$

Since t_1 and t_2 are also unknowns, we should minimize S with respect to α, β, t_1 and t_2. Taking partial derivative of (17) using Liebniz' rule with respect to these variables and equating them to zero, we will have,

$$\frac{\partial S}{\partial \alpha} = -\int_0^{t_1} dx + \int_{t_1}^{t_2} dx - \int_{t_2}^{t_1} dx = 0 \tag{18}$$

$$\frac{\partial S}{\partial \beta} = -\int_0^{t_1} dx + \int_{t_1}^{t_2} dx - \int_{t_2}^{t_1} dx = 0 \tag{19}$$

$$\frac{\partial S}{\partial t_1} = 2\left[y(t_1) - \alpha - \beta t_1\right] = 0 \tag{20}$$

$$\frac{\partial S}{\partial t_2} = -2\left[y(t_2) - \alpha - \beta t_2\right] = 0 \tag{21}$$

Equations (18) through (21) may be solved simultaneously for α, β, t_1 and t_2. Thus, we have the following system of equations,

$$2t_2 - 2t_1 - 1 = 0 \tag{22}$$

$$t_2^2 - t_1^2 - ½ = 0 \tag{23}$$

$$y(t_1) - \alpha - \beta t_1 = 0 \tag{24}$$

$$y(t_2) - \alpha - \beta t_2 = 0 \tag{25}$$

The solutions are:

$$t_1=1/4 \tag{26}$$

$$t_2=3/4 \tag{27}$$

$$\alpha = y(3/4)-(3/4)\beta = y(1/4)-(1/4)\beta \tag{28}$$

$$\beta = 2[y(3/4)-y(1/4)] \tag{29}$$

This procedure may be expanded to include "m" unknown parameters. Some computational methods for solving the different cases of m parameters model are investigated by Ptak (1958), Rice and White (1964), Rice (1964a,b,c,69,85), Usow (1967a), Lazarski (1975a,b,c,77) (see also, Hobby and Rice (1965), Kripke and Rivlin (1965), Watson (1981)). Now, let us have a look at Lorenz curve and its proposed functional forms.

5. LORENZ CURVE

The Lorenz curve for a random variable with probability density function f(v) may be defined as the ordered pair.

$$(P(V|V \le v), \frac{E(V|V \le v)}{E(V)}) \qquad v \in R \tag{30}$$

For a continuous density function f(v), (30) can be written as,

$$(\int_{-\infty}^{v} f(w)dw, \frac{\int_{-\infty}^{v} wf(w)dw}{\int_{-\infty}^{+\infty} wf(w)dw}) \equiv (x(v), y(x(v))) \tag{31}$$

Taguchi (1972a,b,c,73,81,83,87,88) multiplies the second element of (30) by $P(V|V \le v)$ which is not correct; his definition of (31) is equivalent to ours. We denote (31) by ordered pair (x(v),y(x(v))) where x(v) and y(x(v)) are its elements. "x" is a function which maps "v" to x(v) and "y" is a function which maps x(v) to y(x(v)). The function y(x(v)) is simply the Lorenz curve function. For the explicit function for the Lorenz curve, we use the form introduced by Gupta (1984) and a modified version, which benefits from certain properties.

Gupta (1984) proposed the functional form,

$$y = xA^{x-1} \qquad A>1 \tag{32}$$

The modified version of Bidabad and Bidabad (1989) suggests the following functional form:

$$y = x^{B}A^{x-1} \qquad B \ge 1,\ A \ge 1 \tag{33}$$

To estimate the above functions we need discrete data from the population, to construct relevant x and y vectors to estimate "A" of (32) or "A" and "B" of (33). On the other hand if the probability distribution of income is known, we can estimate the Lorenz curve by using the continuous L_1 norm smoothing method for continuous functions.

In the following section we proceed to apply this method to estimate the parameters "A" of (32), and "A" and "B" of (33) by using the information of probability density function of income.

6. CONTINUOUS L_1 NORM SMOOTHING OF LORENZ CURVE

To estimate the Lorenz curve parameters when income probability density function is known, we cannot always take straightforward steps. When the probability density function is easily integrable, there is no major problem in advance. We can find the functional relationship between the two elements of (31) by simple mathematical derivation. However, when integrals of (31) are not obtainable, another procedure is to be adopted.

Suppose that income of a society is distributed with probability density function f(w). This density function may be a skewed function such as Pareto or log-normal, as follows

$$f(w) = \theta k^{\theta} w^{-\theta-1}, \qquad w,k>0,\ \theta>0 \tag{34}$$

$$f(w) = [1/w\sigma\sqrt{(2\pi)}]\exp\{-[\ln(w)-\mu]^2/2\sigma^2\}, \quad w\varepsilon(0,\infty),\ \mu\varepsilon(-\infty,+\infty),\ \sigma>0 \tag{35}$$

These two distributions are known as good candidates for representing distribution of personal income. In the case of Pareto density function of (34), we can simply derive the Lorenz curve function as follows. Let F(w) denote the Pareto distribution function:

$$F(w) = 1-(k/w)^{\theta} \tag{36}$$

With mean equal to,

$$E(w) = \theta^{k}/(\theta-1), \quad \theta>1 \tag{37}$$

If we find the function y as stated by (31) as a function of x, the Lorenz function will be derived. Rearrange the terms of (31) as,

$$x(v) = \int_{-\infty}^{v} f(w)dw \tag{38}$$

$$y(x(v)) = [1/E(W)]\int_{-\infty}^{v} wf(w)dw \quad (39)$$

Substituting Pareto distribution function,

$$x(v) = F(v) = 1-(k/v)^{\theta} \quad (40)$$

$$y(x(v)) = [(\theta-1)/\theta^{k}] \int_{k}^{v} w\theta k^{\theta} w^{-\theta-1} dw \quad (41)$$

Or,

$$y(x(v)) = 1-(k/v)^{\theta-1} \quad (42)$$

By solving (40) for "v" and substituting in (42), the Lorenz curve for Pareto distribution is derived as,

$$y = 1-(1-x)^{(\theta-1)/\theta} \quad (43)$$

For log-normal distribution, we proceed as follows:

As it was shown in the case of Pareto distribution, formula of Lorenz curve is easily obtained. However, if we select the log-normal density function (35), the procedure may not be the same. Because the integral of log-normal function has not been derived yet. In the following pages, the L_1 norm smoothing technique will be developed to estimate the parameters of given functional forms (32) and (33) by using the continuous probability density function.

According to (30) and (31) independent and dependent variables of (32) and (33) may be written as,

$$x(v) = \int_{o}^{v} f(w)dw \quad (44)$$

$$y(x(v)) = [1/E(W)]\int_{0}^{v} wf(w)dw \quad (45)$$

Substitute (44) and (45) in (32) and including random error term u, we'll have,

$$[1/E(w)]\int_{0}^{v} wf(w)dw = \int_{0}^{v} f(w)dw.\, A^{\int_{0}^{v} f(w)dw-1}.e^{u} \quad (46)$$

Or,

$$y(x) = x\, A^{x-1} e^{u} \quad (47)$$

Similarly for the model (35),

$$[1/E(w)]\int_{0}^{v} wf(w)dw = \left\{\int_{0}^{v} f(w)dw\right\}^{B}.\, A^{\int_{0}^{v} f(w)dw-1}.e^{u} \quad (48)$$

Or,

$$y(x) = x^{B} A^{x-1} e^{u} \quad (49)$$

Taking natural logarithm of (47) and (49), gives,

$$\ln y(x) = \ln x + (x-1)\ln A + u \quad (50)$$

$$\ln y(x) = B.\ln x + (x-1)\ln A + u \quad (51)$$

With respect to properties of Lorenz curve and probability density function of f(w) and equations (46) to (49), it can be seen that x belongs to the interval [0,1]. Thus the L_1 norm objective function for minimizing (50) or (51) is given by,

$$\underset{A}{Min}: S = \int_{0}^{1} |u| dx \quad (52)$$

Or,

$$\underset{A}{Min}: S = \int_{0}^{1} |\ln y(x) - \ln x - (x-1)\ln A| dx \quad (53)$$

Or,

$$\underset{A}{Min}: S = \int_{0}^{1} |x-1| \; |[\ln y(x) - \ln]/(x-1) - \ln A| dx \quad (54)$$

By a similar technique used by (9), we can rewrite (54) as,

$$\underset{A}{Min}: S = \int_{0}^{t} |x-1| \{[\ln y(x) - \ln x]/(x-1) - \ln A\} dx - \int_{t}^{1} |x-1| \{[\ln y(x) - \ln]/(x-1) - \ln A\} dx \quad (55)$$

Since, $0 \le x \le 1$ we have,

$$\underset{A}{Min}: S = \int_0^1 \left[\ln y(x) - \ln x - (x-1)\ln A\right]dx + \int_t^1 \left[\ln y(x) - \ln x - (x-1)\ln A\right]dx \tag{56}$$

Differentiate (56) partially with respect to "t" and "A":

$$\frac{\partial S}{\partial A} = +\int_0^1 \left[(x-1)/A\right]dx - \int_t^1 \left[(x-1)/A\right]dx = 0 \tag{57}$$

$$\frac{\partial S}{\partial t} = -2\left[\ln y(t) - \ln t - (t-1)\ln A\right] = 0 \tag{58}$$

From equation (57), we have,

$$t = 1 \pm \sqrt{2}/2 \tag{59}$$

Since "t" should belong to the interval [0,1], we accept,

$$t = 1 - \sqrt{2}/2 \tag{60}$$

Substitute (60) in (58), and solve for "A", gives the L_1 norm estimation for "A" equal to,

$$A = \left[\frac{1-\sqrt{2}/2}{y(1-\sqrt{2}/2)}\right]^{\sqrt{2}} \tag{61}$$

Now, let us apply this procedure to another Lorenz curve functional form of (33) (redefined by (51)). Rewrite L1 norm objective function (52) for the model (51),

$$\underset{A,B}{Min}: S = \int_0^1 \left|\ln y(x) - B\ln x - (x-1)\ln A\right|dx \tag{62}$$

Or,

$$\underset{A,B}{Min}: S = \int_0^1 |x-1|\left|\left[\ln y(x)\right]/(x-1) - (\ln x)/(x-1) - \ln A\right|dx \tag{63}$$

The objective function (63) is similar to (16). Thus, by a similar procedure to those of (17) through (29) we can write "S" as,

$$\begin{aligned}\underset{A,B}{Min}: S = &\int_0^{t_1} |x-1|\left\{\left[\ln y(x)\right]/(x-1) - (\ln x)/(x-1)\ln A\right\}dx \\ &- \int_{t_1}^{t_2} |x-1|\left\{\left[\ln y(x)\right]/(x-1) - (\ln x)/(x-1) - \ln A\right\}dx \\ &+ \int_{t_1}^{1} |x-1|\left\{\left[l y(x)\right]/(x-1) - (\ln x)/(x-1) - \ln A\right\}dx\end{aligned} \tag{64}$$

Since $0 \le x \le 1$, (64) reduces to,

$$\begin{aligned}\underset{A,B}{Min}: S = &-\int_0^{t_1} \left[\ln y(x) - B\ln x - (x-1)\ln A\right]dx + \int_{t_1}^{t_2} \left[\ln y(x) - B\ln x - (x-1)\ln A\right]dx \\ &- \int_{t_2}^{1} \left[\ln y(x) - B\ln - (x-1)\ln A\right]dx\end{aligned} \tag{65}$$

Differentiate "S" partially with respect to "A", "B", t_1 and t_2,

$$\frac{\partial S}{\partial A} = \frac{1}{A}\left[\int_0^{t_1} (x-1)dx - \int_{t_1}^{t_2} (x-1)dx + \int_{t_2}^{1} (x-1)dx\right] = 0 \tag{66}$$

$$\frac{\partial S}{\partial B} = \int_0^{t_1} \ln(x)dx - \int_{t_1}^{t_2} \ln(x)dx + \int_{t_2}^{1} \ln(x)dx = 0 \tag{67}$$

$$\frac{\partial S}{\partial t_1} = -2\left\{\ln\left[y(t_1)\right] - B\ln(t_1) - (t_1 - 1)\ln(A)\right\} = 0 \tag{68}$$

$$\frac{\partial S}{t_2} = 2\{\ln[y(t_2)] - B\ln(t_2) - (t_2 - 1)\ln(A)\} = 0 \quad (69)$$

The above system of simultaneous equations can be solved for the unknowns: t_1, t_2, "A" and "B". Equation (66) is reduced to,

$$t_1^2 - t_2^2 - 2(t_1 - t_2) - 1/2 = 0 \quad (70)$$

Equation (67) can be written as,

$$t_1(\ln t_1 - 1) - t_2(\ln t_2 - 1) - 1/2 = 0 \quad (71)$$

Calculate t_1 from (70) as,

$$t_1 = 1 \pm \sqrt{(t_2^2 - 2t_2 + 3/2)} \quad (72)$$

Since $0 \le t_1 \le 1$ we accept

$$t_1 = 1 - \sqrt{(t_2^2 - 2t_2 + 3/2} \quad (73)$$

Substitute t_1 from (73) into (71), and rearrange the terms, gives;

$$\ln \frac{\left[1 - \sqrt{(t_2^2 - 2t_2 + 3/2)}\right]^{\left[1 - \sqrt{(t_2^2 - 2t_2 + 3/2)}\right]}}{t_2^{t_2}} + t_2 - 3/2 + \sqrt{(t_2^2 - 2t_2 + 3/2)} = 0 \quad (74)$$

The root of equation (74) may be computed by a numerical algorithm. However, it has been computed and rounded for five digits decimal point as,

$$t_2 = 0.40442 \quad (75)$$

Value of t_1 is derived by substituting t_2 into (73);

$$t_1 = 0.07549 \quad (76)$$

Values of "B" and "A" are computed from (68) and (69) using t_2 and t_1 given by (75) and (76). Thus,

$$B = \frac{(t_2-1)\ln y(t_1) - (t_1-1)\ln y(t_2)}{(t_2-1)\ln(t_1) - (t_1-1)\ln(t_2)} \quad (77)$$

Or,

$$B = -0.84857 \ln[y(0.07549)] + 1.31722 \ln[y(0.40442)] \quad (78)$$

And,

$$A = [y(0.07549)]^{1.28986}[y(0.40442)]^{-3.68126} \quad (79)$$

Now, let us describe how equation (61) for the model (32) and equations (78) and (79) for the model (33) can be used to estimate the parameters of the Lorenz curve when the probability distribution function is known. For the model (32) we should solve (44) for $x(v) = 1 - \sqrt{2}/2$. On the other hand, we should find value of "v" such that,

$$x(v) = \int_0^v wf(w)dw = 1 - \sqrt{2}/2 \quad (80)$$

By substituting this value of "v" into (45), value of $y(1 - \sqrt{2}/2)$ is computed. This value is used to compute the parameter "A" given by (61) for model (32).

The procedure for the model (33) is also similar, with the difference that two values of "v" should be computed. Once two different values of "v" are computed as follow,

$$x(v) = \int_0^v wf(w)dw = 0.07549 \quad (81)$$

$$x(v) = \int_0^v wf(w)dw = 0.40442 \quad (82)$$

Values of "v" are substituted in (45) to find y(0.07549) and y(0.40442). These values of "y" are used to compute the parameters of the model (33) by substituting them into (78) and (79).

The computation of related definite integrals of x(v) defined by (80), (81) and (82) can be done by appropriate numerical methods.

7. NUMERICAL EXAMPLE

Suppose the sample mean and median of income distribution of the society are given. For calculation of the parameters of Lorenz curve, the following notations have been coded for MathCAD 11.
Assume that the sample mean of income distribution of the society is: $60,000.
Assume that the sample median of income distribution of the society is: $40,000.

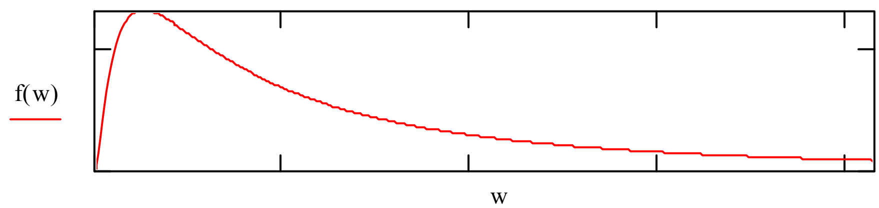
The standard deviation can be calculated as $\sigma := \sqrt{2 \cdot \ln\left(\frac{Mean}{Med}\right)}$

And $\mu = \ln(Med)$ such that $\mu = 10.5966$, $\sigma = .9005$
Calculation of Log-Normal density function parameters based on sample mean and median

Log-Normal Probability Density Function $f(w) := \left(\frac{1}{w \cdot \sigma \cdot \sqrt{2 \cdot \pi}}\right) \cdot \exp\left[\frac{-(\ln(w) - \mu)^2}{2 \cdot \sigma^2}\right]$

Selective range for Log-Normal plot: $w := 10^{-5}, \frac{Mean}{200} .. 2 \cdot Mean$

Figure 2: Log-Normal plot



Precision Tolerance level TOL :=0.00001
TOL value might be changed for more accurate solutions, (less TOL = higher precision)

For equation (45) we have $y(v) := \left(\frac{1}{Mean}\right) \cdot \int_{0}^{v} w \cdot f(w)\, dw$

For equation (44) we have $x(v) := \int_{0.00001}^{v} f(w)\, dw$

Calculation for Gupta model:
Initial guess for v: $v := 20000$ It might be changed for faster convergence and less iterations

For (60) $t_0 := 1 - \frac{\sqrt{2}}{2}$

Calculating v for (80) $v := \mathrm{root}(x(v) - t_0, v),$ $v = 27136.6437$

y(t) $y(v) = 0.04208$ $z_0 := y(v)$

For (61) estimated A: $A := \left(\frac{t_0}{z_0}\right)^{\sqrt{2}}$ $A = 15.54768$

For (53) $S := \int_{0}^{1} \left|\ln(z_0) - \ln(t_0) - (t_0 - 1) \cdot \ln(A)\right|\, dx$

Sum of absolute residuals $S = 0$
Range variable for plotting the Lorenz curves $X := 0, 0.005 .. 1$

Gupta Lorenz curve: $Y(X) := X \cdot A^{X-1}$

Calculation of Gini coefficient $\quad Gini := 1 - 2\cdot\int_0^1 Y(X)\,dX \quad Gini = 0.51967$

<u>Calculation For Bidabad Model:</u>

For (76) $\quad t_1 := 0.07549$

Initial guess for v: $v := 8000$ It might be changed for faster convergence and less iterations

Calculating v for (81) $\quad v := root(x(v) - t_1, v) \quad v = 9464.04318$

y(0.07549) $\quad y(v) = 0.00442 \quad z_1 := y(v)$

For (75) $\quad t_2 := 0.40442$

Initial guess for v: $v := 2700($ It might be changed for faster convergence and less iterations

Calculating v for (82) $\quad v := root(x(v) - t_2, v) \quad v = 38826.25803$

y(0.40442) $\quad y(v) = 0.07722 \quad z_2 := y(v)$

For (79) $\quad A := (z_1)^{1.28986}\cdot(z_2)^{-3.68126}$

For (78) $\quad B := -0.84857\cdot\ln(z_1) + 1.31722\cdot\ln(z_2)$

Estimated A and B: $\quad A = 11.41481 \quad B = 1.22709$

For (62) $\quad S := \int_0^1 \left|\ln(z_1) - B\cdot\ln(t_1) - (t_1 - 1)\cdot\ln(A)\right|\,dx$

Sum of absolute residuals $\quad S = 0.00002$

Range variable for plotting the Lorenz curves $\quad X := 0, 0.005 .. 1$

Modified Lorenz curve $\quad Y(X) := X^B\cdot A^{X-1}$

Calculation of Gini coefficient $\quad Gini := 1 - 2\cdot\int_0^1 Y(X)\,dX \quad Gini = 0.51834$

<u>8.IMPLIED-INEQUALITY-INDEX</u>

Most inequality indices concentrate on statistical aspect of the distribution of income. That is they generally analyze the distribution without inferring about the amount of fund needed to correct income inequality. In this section, we will introduce an inequality index, which shows how much money should be transferred from upper income group to the lower group to achieve the desired distribution of income. Suppose there is a personal income υ at which half of the total income of the population belongs to those who have less than υ and the other half of the income belongs to those who have higher income than υ. That is:

$$\int_{-\infty}^{v} wf(w)dw = \int_{v}^{+\infty} wf(w)dw \tag{83}$$

By definition, we have:

$$\mu = \int_{-\infty}^{+\infty} wf(w)dw = \int_{-\infty}^{v} wf(w)dw + \int_{v}^{+\infty} wf(w) \tag{84}$$

That is:

$$\int_{-\infty}^{v} wf(w)dw = \mu/2 \tag{85}$$

On the other hand:

$$\frac{\int_{-\infty}^{v} wf(w)dw}{\int_{-\infty}^{+\infty} wf(w)dw} = 1/2 \tag{86}$$

According to (31) this is a point on Lorenz curve with the following ordered pair:

$$(\int_{-\infty}^{v} wf(w)dw, 1/2) \tag{87}$$

Thus, we define implied-inequality-index (iii) as $\int_{-\infty}^{v} f(w)dw$ when υ satisfies (83). That is,

$$iii = \int_{-\infty}^{v} f(w)dw \quad \text{when v satisfies} \quad \frac{\int_{-\infty}^{v} wf(w)dw}{\int_{-\infty}^{+\infty} wf(w)dw} = 1/2 \tag{88}$$

To find iii, (85) should be solved for υ and its value be replaced in (88). As iii approaches ½, distribution becomes more symmetric. If iii tends to 1, distribution tends to be fully right-skewed indicating high (right) inequality and as iii tends to 0, distribution tends to be left-skewed and distribution tends to left high inequality. The values of iii less than ½ however have no economic implication for income distribution. Let us define the cost of equalization as:

C= [iii-½]×N×μ (89)

The above expression means that to equalize the distribution of income without changing the average income of the society, the amount of C should be transferred from higher income earner to lower income earner, where N and μ are the population size and average income of the society.

We may normalize this index by dividing the equalization cost by total income of the society and find an inter-societies comparable index. That is:

Relative cost of equalization = [(iii-½)×N×μ] / (N×μ) = (iii-½) (90)

9.NUMERICAL EXAMPLE

To illustrate, the following table 1 of income distribution for a hypothetical society is used. Consider a society of 400 households with total income of the society equal to \$2000 where 280 poor income earners receive half of it (\$1000) and 120 richer earn another 50% (\$1000) of the society's income. These values can be simply understood from table 1. At the half of total income of the society (\$1000) the bottom 70% of the population earns 50% of society's income and 30% of the top of the population earn other 50% of the total income of the society. According to table 1, we have:

N = 400 (Number of households)
υ = μ = 2000/400=5 (Average income)
μ_{lower} = 1000/280 = 3.57 (Average income of lower category)
μ_{upper} = 1000/120 = 8.33, (Average income of upper category)
iii = 280/400 = 0.7 (implied inequality index)
C = (0.7-0.5) × 400×5= \$400 (Cost of equalization)

That is, if we collect total tax of \$400 from the top 30% of the population and transfers it to the lower 70% of the income earners, the average income of both groups will be the same:

(1000+400)/280 = (1000-400)/ 120 = 5
Relative cost of equalization = 0.7-0.5 = 0.2

That is the cost of such equalization is 20% of the total income of the society.

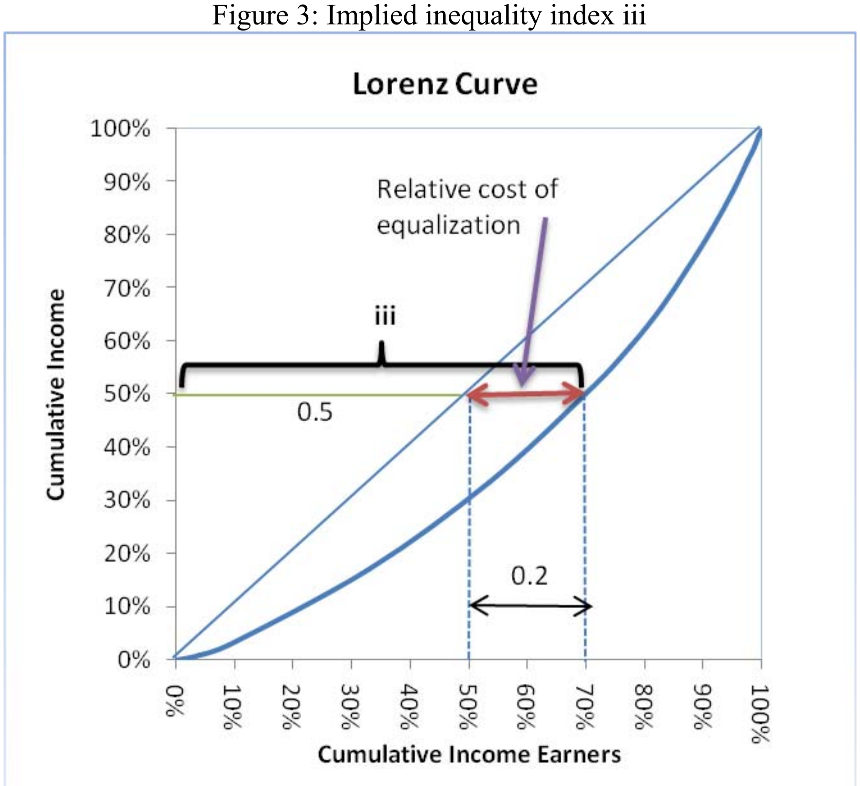
Table 1: Income distribution for a hypothetical society

Income w	Frequency f	Cumulative Frequency F	Relative Frequency	Relative Cumulative Frequency	Half Income Earner	w . f (1)*(2)	Cumulative Income	Relative Cumulative Income	Half Income
($)	(Numbers)	(Numbers)	(%)	(%)	(Numbers)	($)	($)	($)	($)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
1	17	17	4.3%	4.3%		17	17	0.9%	
2	20	37	5.0%	9.3%		40	57	2.9%	
3	95	132	23.8%	33.0%		285	342	17.1%	
4	82	214	20.5%	53.5%		328	670	33.5%	
5	66	280	16.5%	70.0%	280	330	1000	50.0%	1000
6	30	310	7.5%	77.5%		180	1180	59.0%	
7	21	331	5.3%	82.8%		147	1327	66.4%	
8	18	349	4.5%	87.3%		144	1471	73.6%	
9	17	366	4.3%	91.5%		153	1624	81.2%	
10	14	380	3.5%	95.0%		140	1764	88.2%	
11	11	391	2.8%	97.8%		121	1885	94.3%	
12	4	395	1.0%	98.8%		48	1933	96.7%	
13	3	398	0.8%	99.5%		39	1972	98.6%	
14	2	400	0.5%	100.0%	120	28	2000	100.0%	1000
	400		100%		400	2000			2000

In table 1, the column (1) depicts dollar values of income categories and the column (2) shows the number or frequencies of households in each income category of column (1). Columns (3), (4) and (5) are for cumulative frequencies, relative frequencies, and relative cumulative frequencies. Column (6) shows the number of lower and higher income earners. Column (7) shows the multiplication of the paired elements of the columns (1) and (2). Column (8) cumulates (7) and (9) shows the relative cumulative income. The column (10) shows the half of the total income of the society.

According to the table 1, we may depict the iii and the relative cost of equalization on the Lorenz curve as follows. This is depicted by using columns (4) and (9) of the above table.

Figure 3: Implied inequality index iii



This figure depicts the information of table 1. The implied inequality index (iii) and relative cost of equalization are shown as corresponding parts of Lorenz curve

CONCLUSION

Estimation of the Lorenz curve is confronted with some difficulties. To avoid this, we try to estimate the functional form of the Lorenz curve by using continuous information. We used the probability density function of population income to estimate the Lorenz function parameters by the continuous L_1 norm smoothing method. To have a better understanding and policy arrangements about the inequality of income distribution, it is not enough to know the conventional inequality indices. The redistribution policies need to deal with specific budget guidelines to promote the society to a more equal position. Obviously, there is no single "best" measure of income inequality. While there are alternative methods, there is no best way to calculate the inequality index specially concentrating on fiscal view. That is they generally analyze the distribution without inferring about the amount of fund needed to correct income inequality. The viewpoint of this paper is to introduce an implied inequality index, which satisfies these policy implications needs. We designed an implied inequality index as a fiscal guidepost for equalization of society's income.

We did not develop the model to other policy objects. That is, instead of benchmarking of half income of the society we may adopt quantiles or deciles points as equalization policy object. These developments will improve the policy applications of the derived indices.

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